Advanced Topics Winter Trimester DE Project

Directions: You are to prepare a report detailing the following four questions. If a question involves a computation, this should be placed at the end of the report as an appendix. Each computation should be its own appendix (so you should have multiple appendices) and clearly labeled. In the report, when detailing the solution to a problem, clear references to the appropriate appendix should be given. The report itself should be no more than two pages (excluding appendices) and should be written in a narrative structure (do not refer to question numbers or restate the questions).

1. In this problem, we will explore the concept of a retirement account under idealized mathematical conditions. Like the interest rate problems you explored in Pre-Calculus, a retirement account will behave like exponential growth or decay provided the interest on the account is compounded continuously. However, we now need to take the monthly deposit payments you make into account. If we also assume the payment is done continuously, and there is no change in the monthly deposit or interest rate, then the retirement account obeys the IVP

$$\frac{dS}{dt} = rS + k, \quad S(0) = 0$$

where S is the balance, r is the annual interest rate, k is the monthly payment times 12, and t is measured in years.

   a) Retirement accounts invest your money into the stock market. The historical annual rate of return on the S and P 500 is 8.9%. Solve the above differential equation if you assume a return of 8.9%. Your answer should involve k, the monthly payment you make to the account.
   b) Assume you start this savings account at the age of 30. How much money would be in your account at the current retirement age of 67 if you decide to contribute $\$100$ a month? Compute how much money you contribute directly to the account and how much comes from interest earned.
   c) Repeat $b)$ but for monthly contributions of $\$150$, $\$200$, and $\$250$.
   d) The standard rule of thumb for retiring is you need an annual income in retirement that is 80% of your previous income to maintain the same living standard. Pick a profession you can see yourself choosing in the future and research the median salary for that job. What would you need in retirement to maintain the same living standard? How much money do you need in your retirement account at 67 so you can live off the interest alone?
   e) Do any of the monthly contributions in parts $b)$ and $c)$ generate a retirement account with enough money? What is the minimum monthly deposit you need to make to ensure you have enough? Compute how much money you contribute directly to the account with that monthly deposit versus how much comes from interest earned.
   f) Repeat steps $a) - e)$ but now assume you start the retirement account at age 35. How much more money do you need to contribute each month to reach your retirement goal of 80% of pre-retirement income coming solely from interest than if you started the account at 30? How does the accrued interest in the two accounts compare?

g) Repeat steps $a) - e)$ but now assume you start the retirement account at age 25. How much less do you need to contribute each month than if you started at age 30? How does the accrued interest in the two accounts compare?

2. In this question, we will explore the concept of chaos in differential equations. Mathematically, a DE is chaotic if small differences in initial conditions, such as those resulting from measuring or rounding errors, can lead to widely different outcomes for the solutions.

   a) We begin by considering the DE
   $$y' = a - y^2$$
   where a is a parameter. For $a = -1$, $a = 0$, and $a = 1$, solve this DE using separation of variables. How do these solutions compare to each other? (Note: $\int \frac{1}{1-y^2}dy = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right|$)

   b) An equilibrium solution to a DE is a constant solution. These would correspond to $y' = 0$ in the DE itself. How many equilibrium solutions does this DE have if $a = -1$, $a = 0$, and $a = 1$?

   c) A bifurcation diagram is created using the parameter a as the horizontal axis, y as the vertical axis, and then plotting the equilibrium solutions as a function of a. Create a bifurcation diagram for this DE. This kind of bifurcation is called a saddle-node bifurcation.

   d) Repeat step $c)$ with the DE
   $$y' = ay - y^3$$
   This type of bifurcation is called pitchfork bifurcation. Looking at the graph you created, does this name seem appropriate?

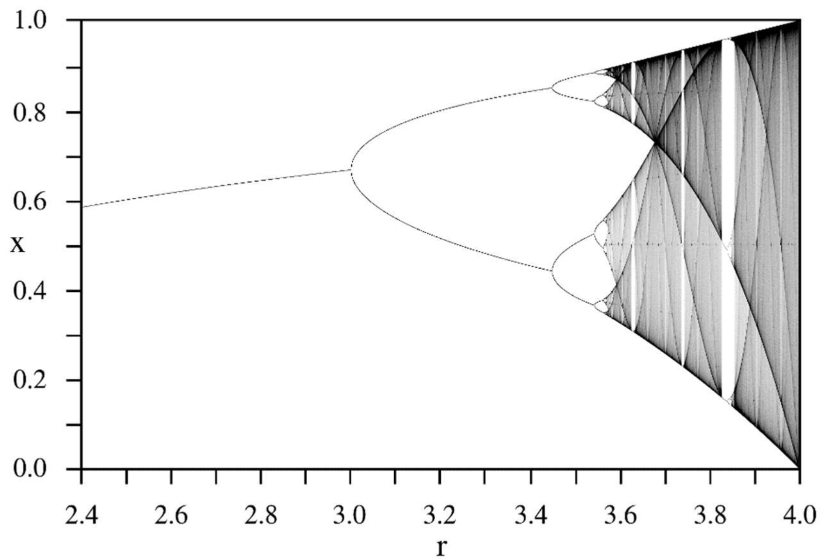
3. We will build upon question 3 by considering the discrete logistic equation. The discrete logistic equation is a difference equation given by
$$x_{n+1} = rx_n(1 - x_n),\ \ x_0\ given$$
Difference equations are like differential equations and several of the same techniques work for both types of equations. The major difference between the two is that the independent variable for a difference equation is the number of iterations and therefore only takes in non-negative integer values. An equilibrium for the discrete logistic equation is a fixed point, that is a point x_n satisfying
$$x_n = rx_n(1 - x_n)$$
A 2-cycle is when the solution does not have an equilibrium but instead bounces back and forth between two values. Similarly, an n-cycle will hit n points in a row before repeating the values.

   a) Open the accompanying Excel spreadsheet and go to sheet 1. This sheet has the difference equation set up with the ability to adjust the r value to be between 0 and 2.9 with a fixed initial condition of 0.7. What do you notice happens to the solution as you adjust the r value?

   b) Adjust the initial condition of $x_0 = 0.7$ to some other value by changing the value in cell C16 (pick a value less than $x_0 = 1$). What did you change it to? Why would I tell you to pick a value less than 1?

   c) Compare and contrast the solutions you get for an initial condition of $x_0 = 0.7$ and the initial condition you picked for a couple different r values. What impacts do you think the initial value has on the solution?

d) Open sheet 2 now. The initial condition is set to $x_0 = 0.7$. Repeat question $a)$ on sheet 2.
e) Repeat $c)$ with your adjusted initial condition on sheet 2.
f) What do you notice about increasing the r value?
g) Open sheet 3. Describe the dynamics that occur as you increase r even further than in problem $e)$. Do you see any patterns arising?
h) Here is the bifurcation diagram for the discrete logistic equation. The bifurcation diagram gives a plot of all equilibrium or cycle values. We plot these values against r:



As r increases, what can you say about solutions? Play around on the spreadsheet for values of r close to values where the bifurcation diagram splits. Do you think small measurement or rounding errors might cause a problem? Give examples via screenshot of the spreadsheet.

4. Reflect on what you have done in this project. What did you learn?